Existence of Explosive Solutions to Non-Monotone Semilinear Elliptic Equations

THESIS

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Abstract

We consider the semilinear elliptic equation $\Delta u = p(x)f(u)$ on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, where f is a nonnegative function which vanishes at the origin and satisfies $g_1 \leq f \leq g_2$ where g_1, g_2 are nonnegative, nondecreasing functions which also vanish at the origin, and p is a nonnegative continuous function with the property that any zero of p is contained in a bounded domain in Ω such that p is positive on its boundary. For Ω bounded, we show that a nonnegative solution u satisfying $u(x) \to \infty$ as $x \to \partial\Omega$ exists provided the function $\psi(s) \equiv \int_0^s f(t)\,dt$ satisfies $\int_1^\infty [\psi(s)]^{-1/2}\,ds < \infty$. For Ω unbounded (including $\Omega = \mathbb{R}^n$), we show that a similar result holds where $u(x) \to \infty$ as $|x| \to \infty$ within Ω and $u(x) \to \infty$ as $x \to \partial\Omega$ if $p(x)$ decays to zero rapidly as $|x| \to \infty$.

Existence of Explosive Solutions to Non-Monotone Semilinear Elliptic Equations

I. Introduction

We consider the semilinear elliptic equation

$$\Delta u \quad = \quad p(x)f(u), \qquad x \in \Omega \subseteq \mathbb{R}^n, \ n \geq 3, \tag{1}$$

where Ω is an open, connected set in $\mathbb{R}^n$ and the function f is nonnegative on $[0, \infty)$ and satisfies the inequality

$$g_1 \leq f \leq g_2, \tag{2}$$

where the functions g_1 and g_2 are continuous and nondecreasing (monotone) on $[0, \infty)$ with $g_1(0) = 0$, $g_2(0) = 0$, $g_1(s) > 0$ and $g_2(s) > 0$ if $s > 0$. We also require the function p to be nonnegative and continuous on $\overline{\Omega}$. We give conditions on the function f which insure that Eq.(1) has a nonnegative solution u for which $u(x) \to \infty$ as $x \to \partial\Omega$. We call these functions explosive (large) solutions of (1) on Ω. If Ω is unbounded, we require $u(x) \to \infty$ as $|x| \to \infty$ within Ω. Furthermore, if $\Omega = \mathbb{R}^n$, we call these functions entire explosive solutions. Such problems arise in the study of steady state diffusion type problems, the study of the subsonic motion of a gas [16], the electric potential in some bodies [13], and Riemannian geometry [6].

Lair [12] gave existence results where the function f was required to be nondecreasing. We note here that our results extend some of those in [12], to the case where f is nondecreasing. In our case we merely require that the function f be nonnegative and f satisfy (2).

For Ω bounded and $p(x) = 1$, Keller [8] and Osserman [15] show that a necessary and sufficient condition for

$$\Delta v_1 = p(x) g_1(v_1), \qquad x \in \Omega \subseteq \mathbb{R}^n, \ n \geq 3, \tag{3}$$

to have an explosive solution is that the function g_1 satisfies

$$\int_1^\infty \left[\int_0^s g_1(t)\, dt \right]^{-1/2} ds < \infty. \tag{4}$$

In Theorem 2.1.2 we show that if (4) holds, then Eq.(1) has a nonnegative explosive solution on a bounded domain Ω. We also show, for $\Omega = \mathbb{R}^n$, that if (4) holds then (1) has a nonnegative entire explosive solution provided p decays to zero rapidly (See Theorem 2.2.2.). Furthermore, we show that if

$$\int_1^\infty \left[\int_0^s g_2(t)\, dt \right]^{-1/2} ds = \infty, \tag{5}$$

then Eq.(1) has no nonnegative explosive solution on a bounded domain Ω (See Theorem 2.1.2.).

Lair [12] has shown that g_1 satisfying (4) is sufficient to guarantee that (3) has a positive explosive solution where the nonnegative, continuous function p is allowed to vanish on large parts of Ω including its boundary. If $g_1(s) = s^\gamma$, then Condition (4) is equivalent to $\gamma > 1$, and the problem of finding nonnegative explosive solutions of Eq.(3) for this particular case, for both Ω bounded and $\Omega = \mathbb{R}^n$, were considered in [6, 11]. Our results will contain these as special cases since our condition on the function f is weaker than that of [12], namely since f, although it is bounded above and below by nonnegative increasing functions, is only required to be nonnegative.

For Ω unbounded, Keller [8], Lair [12], and Osserman [15] give results for a more general f; however, all other known results (except[8, 12, 15]) require f to have the unique forms $f(s) = s^\gamma$, or $f(s) = e^{cs}$. (See, e.g., [5, 6, 11, 13].) Our existence

results, as before, contain these as special cases. We show that, if p decays to zero rapidly as $|x| \to \infty$, then Condition (4) is both necessary and sufficient to guarantee that Eq.(1) has a nonnegative explosive solution on Ω (See Corollary 1.).

1.1 Background

We would like to present preliminary work necessary for our existence results, however, before this we seek a deeper understanding to the origins of Eq.(1). In the work to follow we provide an outline of previous work that, through the years, has led to our problem.

Explosive solutions of semilinear elliptic equations of the type

$$\Delta u = f(u), \qquad x \in \Omega, \quad u|_{\partial\Omega} = \infty$$

were first studied in 1916 for the case $f(u) = e^u$ by Bieberbach [3]. In [3] he had shown that the equation has a unique classical explosive solution in a bounded domain with smooth boundary in $\mathbb{R}^2$. These results were later generalized by J.B Keller [8] and Robert Osserman [15] in 1957. In [8] it was shown that a relatively simple upper bound is obtained for any solution, in any number of variables, of the nonlinear equation $\Delta u = f(u)$. The bound is determined by the function $f(u)$ which, in turn, must be positive and satisfy a particular growth condition G. It turns out that this growth condition is the, now familiar, growth condition given by inequality (4) with g_1 replaced by f. An important result of Theorem 3 in [8] is the existence of explosive solutions, in any bounded domain, to $\Delta u = f(u)$ provided that $f(u)$ is an increasing function. This result, as we will see, is very important as it was later used by Lair [12] who then provided a necessary and sufficient condition for existence of explosive solutions to Eq.(1) where the function f was also required to be nondecreasing.

In [15] the existence of solutions of the nonlinear differential inequality $\Delta u \geq f(u)$ were established in $\mathbb{R}^n$. Osserman showed the condition that the function $f(u)$ be convex was no longer required. Furthermore, he was able to attain greater information on the behavior of solutions.

Cheng and Ni [6] gave a complete classification of all possible solutions of the problem

$$\Delta u = p(x)u^\gamma, \qquad x \in \mathbb{R}^n, \tag{6}$$

where $\gamma > 1$ and the nonnegative function p behaves like $|x|^{-l}$ near infinity for some $l > 2$. In particular, they show that Eq.(6) has a unique entire explosive solution in $\mathbb{R}^n$ that blows up at infinity at the rate of $|x|^q$ where $q = (l-2)/(\gamma-1)$. Furthermore, Bandle and Marcus [2] proved the existence and uniqueness of an explosive positive solution in bounded and unbounded domains (not all of $\mathbb{R}^n$) for the more general equation

$$\Delta u = g(x, u),$$

which includes the case $g(x, u) = p(x)u^\gamma$ where $\gamma > 1$ and $p(x)$ is a positive continuous function in $\overline{\Omega}$ such that p and $1/p$ are bounded. They also studied the behavior of the explosive solutions near the boundary of Ω.

Lair and Wood [11] then further extended the results of Cheng and Ni [6] and Bandle and Marcus [2]. In particular they proved the existence of explosive solutions on a bounded domain Ω with conditions on p relaxed to allow it to be zero on large parts of $\overline{\Omega}$ including $\partial\Omega$. This extended the results of [2, 6] where p was either required to be positive and continuous on $\overline{\Omega}$ (see [2]) or p was required to be positive on $\partial\Omega$ (see [6]). Furthermore, they showed that Eq.(6) has an entire explosive solution under more general conditions on p than given in [6]. In particular, instead

of requiring $|x|^m p(x)$ to be bounded above for some $m > 2$, they simply require

$$\int_0^\infty r\phi(r)\, ds < \infty, \tag{7}$$

where $\phi(r) = \max_{|x|=r} p(x)$. They also proved the existence of explosive solutions to (6) on unbounded domains which are not all of $\mathbb{R}^n$. Furthermore, they prove that no entire explosive solutions of Eq.(6) exist if $0 \le \gamma \le 1$ and p satisfies inequality (7). Lastly, they showed that their positiveness condition on p, which requires any zeros of p be contained in a domain within Ω on whose boundary p is positive, is nearly optimal. In particular, they showed that that if p vanishes in a neighborhood of $\partial\Omega$, then no explosive solution of (6) exists.

All of the previously stated work is what led up to the results of Lair [12]. As stated before, the works of Cheng and Ni [6], Bandle and Marcus [2], and Lair and Wood [11] can be treated as special cases of [12]. With this more in-depth understanding of the origins of Eq.(1) at hand, we now seek a grasp of the underlying elliptic theory necessary for our main results.

1.2 Preliminaries

Before we present our main results, it will be important to present some of the underlying theory in order to form a basis that allows us to prove our results. The first, and probably the most basic is that of upper/lower solutions, which are also referred to as barrier methods. We define them here as they will be used later in proving our main results.

Definition 1.2.1. *An upper solution to the following boundary value problem*

$$\begin{aligned} \Delta u &= p(x)f(u), \qquad x \in \Omega, \\ u(x) &= k, \qquad x \in \partial\Omega, \end{aligned} \tag{8}$$

is a function $\overline{u}$ *satisfying*

$$\begin{aligned} \Delta\overline{u} &\leq p(x)f(\overline{u}) \quad x \in \Omega, \\ \overline{u} &\geq k \quad x \in \partial\Omega. \end{aligned}$$

A lower solution to (6) *is a function* $\underline{u}$ *satisfying*

$$\begin{aligned} \Delta\underline{u} &\geq p(x)f(\underline{u}) \quad x \in \Omega, \\ \underline{u} &\leq k \quad x \in \partial\Omega. \end{aligned}$$

Theorem 1.2.2. *(Theorem* 2.3.1 *of* [17]*) Let* ϕ *be an upper solution and* ξ *a lower solution with* $\xi \leq \phi$ *on* Ω *to Eq.*(8)*. Then there exists a solution* u *to* (8) *with* $\xi \leq u \leq \phi$*.*

We now present the standard maximum principle argument from elliptic theory. Although there are many variations of this principle, we give the theorem which is most useful for our results.

Theorem 1.2.3. *(Theorem 3.3 of* [7]*) Let* L *be a linear elliptic differential operator of the form*

$$Lu = a_{ij}(x)D_{ij}u + b_iD_iu + c(x)u, \qquad a_{ij} = a_{ji},$$

where $x = (x_1, x_2, ..., x_n)$ *in* $\Omega \subseteq \mathbb{R}^n$ *with* $c(x) \leq 0$ *in* Ω*. Suppose that* u *and* v *are functions in* $C^2(\Omega) \cap C(\overline{\Omega})$ *satisfying* $Lu \geq Lv$ *in* Ω *and* $u \leq v$ *on* $\partial\Omega$*, then* $u \leq v$ *in* Ω*.*

Since the laplacian is a linear elliptic differential operator, letting $L = \Delta$ in the above theorem will be of great use in later proving our results. Before moving on we present a very important concept of elliptic theory, namely, gradient estimates. In order to prove our results for the case where $\Omega = \mathbb{R}^n$ we will need the following theorem.

Theorem 1.2.4. *(Theorem 3.9 of* [7]*) Let $u \in C^2(\Omega)$ satisfy Poisson's equation, $\Delta u = f$, in Ω. Then*

$$\sup_{\Omega} d_x |\nabla u(x)| \leq C(\sup_{\Omega} |u(x)| + \sup_{\Omega} d_x^2 |f(x)|),$$

where $C = C(n)$ and $d_x = dist(x, \partial\Omega)$.

From this result we will be able to determine gradient bounds for successive subsequences of approximate solutions to (1) and we will then have that our approximate solutions are equicontinuous, and as long as their solutions are uniformly bounded we may apply the Ascoli-Arzela theorem which guarantees the existence of a convergent subsequence, thus establishing existence. A standard bootstrap argument will then show that our solutions are, in fact, classical solutions of Eq.(1). We now present the Ascoli-Arzela Theorem as it will be crucial in later establishing our existence results.

Definition 1.2.5. *(Definition* 1.17 *of* [1]*) A subset K of a normed space X is called compact if every sequence of points in K has a convergent subsequence in X to an element of K. Furthermore, a subset K of X is called precompact in X if its closure $\overline{K}$ in the norm topology of X is compact.*

Theorem 1.2.6. *(Theorem* 1.34 *of* [1]*)(Ascoli-Arzela Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ if the following two conditions hold:*

(i) There exists a constant M such that $|\phi(x)| \leq M$ holds for every $\phi \in K$ and $x \in \Omega$.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K$, $x, y \in \Omega$, and $|x - y| < \delta$, then $|\phi(x) - \phi(y)| < \varepsilon$.

Before we present the very important theory of Sobolev spaces, we first establish two useful concepts of elliptic theory. We first present a definition of what it means for a bounded domain Ω to have C^2-boundary.

Definition 1.2.7. *A bounded domain $\Omega \subseteq \mathbb{R}^n$ has C^2-boundary if at each point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0, R)$ centered at x_0 with radius R and a one-to-one mapping ω on B onto $\Omega \subseteq \mathbb{R}^n$ such that:*

$$(i)\ \omega(B \cap \Omega) \subset \mathbb{R}^n_+;\ (ii)\ \omega(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+;\ (iii)\ \omega \in C^2(B), \omega^{-1} \in C^2(D).$$

We next would like to establish a definition of the Hölder space $C^{2+\alpha}(\Omega)$. This space is important as we will later show that our solutions are in $C^{2+\alpha}$. However, before we present this definition, we first need an understanding of Hölder continuity.

Definition 1.2.8. *Let x_0 be a point in $\mathbb{R}^n$ and f a function defined on a bounded open set Ω containing x_0. For $0 < \alpha < 1$, we say that f is Hölder continuous with exponent α at x_0 if the quantity*

$$[f]_{\alpha;x_0} = \sup_{x \in \Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. We call $[f]_{\alpha;x_0}$ the α-Hölder coefficient of f at x_0 with respect to Ω. Furthermore, we say f is uniformly Hölder continuous with exponent α in Ω provided the quantity

$$[f]_{\alpha;\Omega} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \quad 0 < \alpha \leq 1,$$

is finite.

Definition 1.2.9. *The Hölder space $C^{2+\alpha}(\Omega)$ is a subspace of $C^2(\Omega)$ consisting of functions whose second order partial derivatives are uniformly Hölder continuous with exponent α in Ω.*

We now present the very important theory of Sobolev spaces that will be crucial in later showing that the standard bootstrap argument can be used to prove that our solutions are classical solutions of Eq.(1) on Ω. As will be demonstrated later, the bootstrap argument makes use of Sobolev imbeddings in order to show that our

solutions are truly $C^{2+\alpha}$ (i.e., that our solutions are sufficiently smooth on Ω). At this point we find it prudent to define first exactly what it means to be an imbedding.

Definition 1.2.10. *(Definition 1.25 of [1]) We say the normed space X is imbedded in the normed space Y, and we write $X \to Y$ to designate this imbedding, provided that*

(i) X is a vector subspace of Y, and
(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

We shall now establish some facts concerning the Sobolev Imbedding Theorem; however, before we present this theorem we first must comprehend the spaces involved.

Definition 1.2.11. *A normed linear space $\mathcal{B}$ is a Banach space provided the space $\mathcal{B}$ is complete. That is, provided every Cauchy sequence converges in $\mathcal{B}$.*

Definition 1.2.12. *Let u be locally integrable in Ω and α a multi-index. Then, a locally integrable function v is called the αth weak derivative of u if it satisfies*

$$\int_\Omega \varphi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \, dx \qquad for\ all\ \varphi \in C_0^{|\alpha|}(\Omega).$$

Furthermore, we call a function k-times weakly differentiable if all its weak derivatives exist for orders up to and including k.

Definition 1.2.13. *A Sobolev space $W^{m,p}(\Omega)$ is a Banach space defined by*

$$W^{m,p}(\Omega) = \{u \in L^p : D^\alpha u \in L^p(\Omega)\ for\ all\ |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$, and the derivatives $D^\alpha u$ are weakly differentiable.

At this point we present our first chain of imbeddings that will be needed for later use in the standard bootstrap argument. From [1] we have the chain of

imbeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega),$$

where $W_0^{m,p}(\Omega)$ is a Sobolev space with compact support, and $L^p(\Omega)$ is the classical Banach space of measurable functions on Ω that are p-integrable, $p \geq 1$. We further define norms in the spaces $W^{m,p}$ and L^p. The norm in the Sobolev space $W^{m,p}$ is given by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u|^p \, dx \right)^{1/p}.$$

We also have the norm in $L^p(\Omega)$, $1 \leq p < \infty$, defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p},$$

and for $p = \infty$ we get

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u|.$$

Notice that the function u can be in $L^\infty(\Omega)$ and still have $\sup_\Omega |u| = \infty$ which is why we require the "essential" supremum.

Definition 1.2.14. *(Definition 4.6 of [1]) The domain Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.*

Definition 1.2.15. *(Definition 4.9 of [1]) The domain Ω satisfies the strong local Lipschitz condition if there exists positive numbers δ and M, a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and, for each j a real-valued function f_j of $n-1$ variables, such that the following conditions hold:*

(i) For some finite R, every collection of $R+1$ of the sets U_j has empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta$ such that $|x - y| \leq \delta$, there exists j such that

$$x, y \in V_j \equiv \{x \in U_j : dist(x, \partial U_j) > \delta\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M: that is, if $\beta = (\beta_1, ..., \beta_{n-1})$ and $\rho = (\rho_1, ..., \rho_{n-1})$ are in $\mathbb{R}^{n-1}$, then

$$|f(\beta) - f(\rho)| \leq M|\beta - \rho|.$$

(iv) For some Cartesian coordinate system $(\zeta_{j,1}, ..., \zeta_{j,n})$ in U_j, $\Omega \cap U_j$ is represented by the inequality

$$\zeta_{j,n} < f_j(\zeta_{j,1}, ..., \zeta_{j,n}).$$

If Ω is bounded, the rather complicated set of conditions above reduce to the simple condition that Ω should have a locally Lipschitz boundary, that is, that each point x on the boundary of Ω should have a neighborhood U_x whose intersection with $\partial\Omega$ should be the graph of a Lipschitz continuous function.

Definition 1.2.16. *We define the space of bounded continuous functions $C_B^j(\Omega)$ to consist of those functions $u \in C^j(\Omega)$ for which $D^\alpha u$ is bounded on Ω for $0 \leq |\alpha| \leq j$. Furthermore, $C_B^j(\Omega)$ is a Banach space with norm given by*

$$\| u \|_{C_B^j(\Omega)} = \max_{0 \leq \alpha \leq j} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

With these definitions, we now present the Sobolev Imbedding Theorem.

Theorem 1.2.17. *(Theorem 4.12 of [1]) Let Ω be a domain in $\mathbb{R}^n$ and, for $1 \leq k \leq n$, let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$. (If $k = n$, the $\Omega_k = \Omega$.) Let $j \geq 0$ and $m \geq 1$ be integers and let $1 \leq p < \infty$.*

PART I *Suppose Ω satisfies the cone condition.*

Case A *If either $mp > n$ or $m = n$ and $p = 1$, then*

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega).$$

Moreover, if $1 \leq k \leq n$, then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad for\ p \leq q \leq \infty,$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad for\ p \leq q \leq \infty.$$

Case B *If $1 \leq k \leq n$ and $mp = n$, then*

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad for\ p \leq q \leq \infty,$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad for\ p \leq q \leq \infty.$$

Case C *If $mp < n$ and either $n - mp < k \leq n$ or $p = 1$ and $n - m \leq k \leq n$, then*

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad for\ p \leq q \leq p^* = kp/(n - mp).$$

In particular,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad for\ p \leq q \leq p^* = np/(n - mp).$$

The imbedding constants for the imbeddings above depend only on n, m, p, q, j, k, and the dimensions of the cone C in the cone condition.

PART II *Suppose Ω satisfies the strong local Lipschitz condition. Then the target space $C_B^j(\Omega)$ of the first imdedding above can be replaced with the smaller space $C^j(\overline{\Omega})$, and the imbedding can be further refined as follows:*

If $mp > n > (m-1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}), \qquad for\ 0 < \lambda \leq m - (n/p),$$

and if $n = (m-1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}), \qquad for\ 0 < \lambda < 1.$$

Also, if $n = m - 1$ and $p = 1$, then the above imbedding holds for $\lambda = 1$ as well.

PART III *All of the imbeddings in Parts A and B are valid for arbitrary domains Ω if the W-space undergoing the imbedding is replaced with the corresponding W_0-space.*

We now have an adequate understanding of the standard elliptic theory necessary to prove our results. Although not all of the previously stated theory will be used explicitly in our results, the derivations should be clear. At this point we are ready to present our main results.

II. Main Results

In this section we state and prove our results. Throughout this thesis, we require the nonnegative function p to satisfy the following *circumferentially positive (c-positive)* condition:

(*c-positive on* Ω) For any $x_0 \in \Omega$ satisfying $p(x_0) = 0$, there exists a domain Ω_0 such that $x_0 \in \Omega_0$, $\overline{\Omega}_0 \subset \Omega$, and $p(x) > 0$ for all $x \in \partial\Omega_0$.

In order to prove our first existence result we first establish the following lemma.

Lemma 2.0.18. *Let* $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$, $n \geq 3$, *and define* $h(r) = (1 + r^2)^{-1/2}$, *where* $r(x) \equiv |x - x_0|$. *Then,* $\Delta h(r) < 0$ *on* $\overline{\Omega}$.

Proof: We have

$$\begin{aligned}
\Delta h(r) &= h'' + \frac{n-1}{r} h' \\
&= 3r^2(1+r^2)^{-5/2} - (1+r^2)^{-3/2} - \frac{n-1}{r} r(1-r^2)^{-3/2} \\
&= (1+r^2)^{-5/2}[3r^2 - (1+r^2) - (n-1)(1+r^2)] \\
&= (1+r^2)^{-5/2}[3r^2 - 1 - r^2 - (n-1) - (n-1)r^2] \\
&= (1+r^2)^{-5/2}[-1 - (n-1) + (3-n)r^2].
\end{aligned}$$

By hypothesis we have that $n \geq 3$. Thus $\Delta h(r) < 0$. This completes the proof.

■

2.1 Existence of Solutions on Bounded Domains

Proposition 2.1.1. *Suppose* Ω *is a bounded domain in* $\mathbb{R}^n$ *with* C^2 *boundary,* p *a nonnegative* $C(\overline{\Omega})$ *function that is c-positive on* Ω, *and* $g_1 \leq f \leq g_2$ *where* g_1

and g_2 are both nonnegative, continuous, and monotone on $[0, \infty)$. Then, for any nonnegative constant c, the boundary value problem

$$\begin{aligned} \Delta v &= p(x)f(v), \quad x \in \Omega, \\ v(x) &= c, \quad x \in \partial\Omega \end{aligned} \tag{9}$$

has a nonnegative classical solution v on Ω.

Proof: From Lair [12] we have that for any nonnegative constant c there exist unique nonnegative classical solutions v_1 and v_2 to the following boundary value problems

$$\begin{aligned} \Delta v_1 &= p(x)g_1(v_1), \quad x \in \Omega, \\ v_1(x) &= c, \quad x \in \partial\Omega, \end{aligned} \tag{10}$$

$$\begin{aligned} \Delta v_2 &= p(x)g_2(v_2), \quad x \in \Omega, \\ v_2(x) &= c, \quad x \in \partial\Omega. \end{aligned} \tag{11}$$

We now wish to show that $v_1 \geq v_2$ on $\overline{\Omega}$. To do this, suppose $v_1 < v_2$ at some point in $\overline{\Omega}$. Since $v_1 = v_2$ on $\partial\Omega$, there must be a point in Ω where $v_1 < v_2$. Now, choose a positive number ε small enough such that $\max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)] > 0$, where $h(r)$ is defined as in Lemma 2.0.18. Then,

$$0 < v_2(x_0) - v_1(x_0) - \varepsilon h(r) \equiv \max_{\overline{\Omega}}[v_2(x) - v_1(x) - \varepsilon h(r)].$$

Thus, at x_0, where the maximum occurs we have

$$\begin{aligned}
0 &\geq \Delta(v_2 - v_1 - \varepsilon h(r)) \\
&= \Delta v_2 - \Delta v_1 - \varepsilon \Delta h(r) \\
&= p(x_0)[g_2(v_2(x_0)) - g_1(v_1(x_0))] - \varepsilon \Delta h(r) \\
&\geq p(x_0)[g_1(v_2(x_0)) - g_1(v_1(x_0))] - \varepsilon \Delta h(r) \\
&\geq -\varepsilon \Delta h(r) \\
&> 0,
\end{aligned}$$

a contradiction. Thus, $v_2 \leq v_1$ in $\overline{\Omega}$.

Now, letting $\overline{v} = v_1$ and $\underline{v} = v_2$ we have that $\underline{v} \leq \overline{v}$ in Ω and

$$\Delta \overline{v} = p(x) g_1(\overline{v}) \leq p(x) f(\overline{v}), \qquad x \in \Omega \tag{12}$$

$$\Delta \underline{v} = p(x) g_2(\underline{v}) \geq p(x) f(\underline{v}), \qquad x \in \Omega \tag{13}$$

Thus, $\overline{v}$ and $\underline{v}$ are upper and lower solutions, respectively, of $\Delta v = p(x) f(v)$, $x \in \Omega$. Hence, by Theorem 1.2.2, Eq.(9) has a nonnegative classical solution v on Ω with $\underline{v} \leq v \leq \overline{v}$.

$\blacksquare$

Theorem 2.1.2. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary and p is a nonnegative $C(\overline{\Omega})$ function which is also c-positive. If the function g_1 satisfies inequality* (4)*, then Eq.(1) has a nonnegative explosive solution in Ω. Furthermore, if*

$$\int_1^\infty \left[\int_0^s g_2(t)\, dt \right]^{-1/2} ds = \infty,$$

then Eq.(1) has no nonnegative explosive solution on Ω.

Proof: Suppose that f satisfies inequality (4) and recall, from hypothesis, that $g_1 \leq f \leq g_2$. From this we have that

$$\int_1^\infty \left[\int_0^s g_2(t)\,dt\right]^{-1/2} ds \leq \int_1^\infty \left[\int_0^s f(t)\,dt\right]^{-1/2} ds \leq \int_1^\infty \left[\int_0^s g_1(t)\,dt\right]^{-1/2} ds < \infty.$$

Let v_k and w_k be the unique nonnegative solutions of

$$\begin{aligned} \Delta v_k &= p(x) g_1(v_k), \quad x \in \Omega, \\ v_k(x) &= k, \quad x \in \partial\Omega, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \Delta w_k &= p(x) g_2(w_k), \quad x \in \Omega, \\ w_k(x) &= k, \quad x \in \partial\Omega. \end{aligned} \tag{15}$$

Then, from [12] we have that the sequence $\{v_k\}$ and $\{w_k\}$ are nondecreasing. We note here that using an analogous upper/lower solution approach to that of Proposition 2.1.1, in order to attain a sequence of solutions $\{u_k\}$ to Eq.(1), will not work here since the desired sequence u_k is not necessarily monotone; which, in turn, implies that although the sequence u_k is bounded, it need not converge. To show that (1) does indeed have a nonnegative explosive solution in Ω we will construct a monotone sequence of solutions analogous to those of (14) and (15). This, as we will see, is not very difficult. To do this, let us first consider the case where $k = 1$ in the above boundary value problems. We have

$$\begin{aligned} \Delta v_1 &= p(x) g_1(v_1), \quad x \in \Omega, \\ v_1(x) &= 1, \quad x \in \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} \Delta w_1 &= p(x)g_2(w_1), \quad x \in \Omega, \\ w_1(x) &= 1, \quad x \in \partial\Omega. \end{aligned}$$

By virtue of Proposition 2.1.1, letting $\overline{u_1} = v_1$ and $\underline{u_1} = w_1$ we have that there exists a nonnegative classical solution u_1 of

$$\begin{aligned} \Delta u_1 &= p(x)f(u_1), \quad x \in \Omega, \\ u_1(x) &= 1, \quad x \in \partial\Omega, \end{aligned}$$

with $w_1 = \underline{u_1} \leq u_1 \leq \overline{u_1} = v_1$. We now consider the following system of equations

$$\begin{aligned} \Delta v_2 &= p(x)g_1(v_2), \quad x \in \Omega, \\ v_2(x) &= 2, \quad x \in \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} \Delta u_1 &= p(x)f(u_1), \quad x \in \Omega, \\ u_1(x) &= 1, \quad x \in \partial\Omega. \end{aligned}$$

In this case, letting $\overline{u_2} = v_2$ and $\underline{u_2} = u_1$ we have that there exists a nonnegative classical solution u_2 of

$$\begin{aligned} \Delta u_2 &= p(x)f(u_2), \quad x \in \Omega, \\ u_2(x) &= 2, \quad x \in \partial\Omega, \end{aligned}$$

with $w_1 \leq u_1 \leq u_2 \leq \overline{u_2} = v_2$. Continuing this line of reasoning we have that there exists a nonnegative classical solution u_k to

$$\begin{aligned}
\Delta u_k &= p(x)f(u_k), \quad x \in \Omega, \\
u_k(x) &= k, \quad x \in \partial\Omega,
\end{aligned}$$

with $w_1 \leq u_{k-1} \leq u_k \leq v_k$, $k \geq 2$. By construction the sequence $\{u_k\}$ is monotone. Furthermore, we already know that the sequence $\{v_k\}$ is monotonic and converges to a classical solution v of

$$\begin{aligned}
\Delta v &= p(x)g_1(v), \quad x \in \Omega, \\
v(x) &\to \infty, \quad x \to \partial\Omega.
\end{aligned}$$

It then follows that $w_1 \leq u_{k-1} \leq u_k \leq v$. Thus, u_k is also bounded above and below. Hence, the sequence $\{u_k\}$ converges on Ω to some function u. We now outline the standard bootstrap argument which proves the function $u(x)$ is indeed a solution to (1).

Let $x_0 \in \Omega \subseteq \mathbb{R}^n$, and $B(x_0, r)$ the ball of radius r centered at x_0 such that it is contained in Ω. Let ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/2)}$ and zero off $B(x_0, r)$. We have

$$\Delta(\psi u_k) = 2\nabla\psi \cdot \nabla u_k + q_k, \quad k \geq 1,$$

where

$$q_k = u_k\Delta\psi + \psi\Delta u_k \tag{16}$$

is a term whose L^∞ norm is bounded independently of k on $B(x_0, r)$. We therefore have

$$\psi u_k \Delta(\psi u_k) = A_k \cdot \nabla(\psi u_k) + s_k, \tag{17}$$

where $A_k = 2u_k \nabla \psi$ and $s_k = \psi u_k q_k - u_k[(2u_k \nabla \psi \cdot \nabla \psi)]$ are bounded independently of k. Now, integrating (17) over $B(x_0, r)$ we have

$$\begin{aligned}
\int_{B(x_0,r)} |\nabla(\psi u_k)|^2 \, dx &= -\int_{B(x_0,r)} [A_k \cdot \nabla(\psi u_k) + s_k] \, dx \\
&\leq \bar{c}_1 \left(\int_{B(x_0,r)} |A_k||\nabla(\psi u_k)| \, dx \right) + c_2 \\
&\leq c_1 \left(\int_{B(x_0,r)} |\nabla(\psi u_k)|^2 \, dx \right)^{1/2} + c_2,
\end{aligned}$$

where $\bar{c}_1$, c_1, and c_2 are some constants independent of k. Hence we have that

$$\| \nabla(\psi u_k) \|^2_{L^2(B(x_0,r))} \leq c_1^2 + 2c_2.$$

From this, it follows that the $L^2(B(x_0, r))$-norm of $|\nabla(\psi u_k)|$ is bounded independently of k. Hence, the $L^2(B(x_0, r/2))$-norm of $|\nabla u_k|$ is bounded independently of k. Similarly, letting ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/4)}$ and zero off $B(x_0, r/2)$, we may show that the $W^{2,2}(B(x_0, r/4))$-norm of u_k is bounded independently of k. It then follows from the Sobolev embedding theory that the $L^q(B(x_0, r/4))$-norm of $|\nabla u_k|$ is bounded independently of k for $q = 2n/(n-2)$.

Continuing this line of reasoning we arrive at a number $r_1 > 0$ such that there is a subsequence of $\{u_k\}_1^\infty$, which we may assume is still the sequence itself, which converges in $C^{1+\alpha}(\overline{B(x_0, r_1)})$, for some positive number $\alpha < 1$.

Let ψ be a C^∞ function equal to 1 on $\overline{B(x_0, r_1/2)}$ and equal to zero off $B(x_0, r_1)$. Then

$$\Delta(\psi u_k) = 2\nabla\psi \cdot \nabla u_k + \hat{q}_k,$$

where $\hat{q}_k$ is given in (16). Now, we consider two cases regarding the regularity of the function $p(x)$.

Case 1: $p(x) \in C^\infty(\Omega)$. The right-hand side of the above equation converges in $C^\alpha(\overline{B(x_0, r_1)})$. Hence, by Schauder theory, $\{\psi u_k\}_1^\infty$ converges in $C^{2+\alpha}(\overline{B(x_0, r_1/2)})$. Since x_0 was arbitrary, it follows that $u \in C^{2+\alpha}(\mathbb{R}^n)$ and hence a solution to Eq.(1)

Case 2: $p(x) \in C(\Omega)$. Since the subsequence $\{u_k\}_1^\infty$ converges in $C^{1+\alpha}(\overline{B(x_0, r_1)})$ we have that $u_k \overset{s-C(B(x_0,r_1))}{\longrightarrow} u$, and consequently $\Delta u_k = p_k(x) f(u_k) \overset{s-C(B(x_0,r_1))}{\longrightarrow} p(x) f(u) \equiv z$. Using the fact that the laplacian is a closed linear operator implies that $u \in D(\Delta)$, and $\Delta u = z$. Furthermore, since x_0 was chosen arbitrarily, we have that u is a classical solution of (1).

We next show that our solution u is an explosive solution. It is easy to show that the function u blows up on $\partial\Omega$ since $\{u_k\}$ is monotone with $u_k = k$ on $\partial\Omega$, we provide the details here. Take any $M > 0$ such that for values of x near $\partial\Omega$ we have $u_k(x) \geq k - 1 > M$. Thus, since our solution $u(x)$ is monotone we have $u(x) \geq u_k(x) \geq k - 1 > M$. It is now clear that u is an explosive solution of (1).

Let us now suppose that $\int_1^\infty \left[\int_0^s g_2(t)\, dt\right]^{-1/2} ds = \infty$ and assume, for contradiction, that u is a nonnegative explosive solution of Eq.(1). Let $\{v_k\}$ be the unique nonnegative classical solution of

$$\begin{aligned} \Delta v_k &= p(x) g_2(v_k), \quad x \in \Omega, \\ v_k(x) &= k, \quad x \in \partial\Omega, \end{aligned} \tag{18}$$

which exists by virtue of Lair (see Proposition 1 of [12]). Then, the sequence $\{v_k\}$ is nondecreasing and one can show (e.g., see Proposition 2.1.1.) that $v_k \leq u$ on Ω. It follows that $\{v_k\}$ converges to a nonnegative function v on Ω. Another standard bootstrap argument will show that v is a classical solution of Eq.(18). Clearly v is also a nonnegative explosive solution on Ω. This, however, cannot happen since from [12] we have that if (5) holds then (18) has no nonnegative explosive solution on Ω. Hence, Eq.(1) has no nonnegative explosive solution on Ω. This completes the proof.

■

2.2 Existence of Solutions on Unbounded Domains

We now consider the case where Ω is unbounded. We commence by considering the case where $\Omega = \mathbb{R}^n$. It is apparent from previous work (see, e.g., [5, 6, 11, 15]) that the function p cannot behave arbitrarily as $|x| \to \infty$ if we anticipate Eq.(1) to have an entire explosive solution. We therefore add an asymptotic condition to the function p and establish results for unbounded Ω similar to those of Theorem 2.1.2. Prior to that, however, we need to prove an inequality that will be needed later.

Lemma 2.2.1. *Suppose g_1 satisfies inequality* (4)*. Then*

$$\int_1^\infty \frac{1}{f(s)}\,ds < \infty. \tag{19}$$

Proof: We first note that since g_1 satisfies inequality (4), then by [12] we have

$$\int_1^\infty \frac{1}{g_1(s)}\,ds < \infty. \tag{20}$$

Using this result, along with inequality (2) we have

$$\int_1^\infty \frac{1}{f(s)}\,ds \leq \int_1^\infty \frac{1}{g_1(s)}\,ds < \infty. \tag{21}$$

Thus inequality (19) holds, completing the proof.

$\blacksquare$

Theorem 2.2.2. *Suppose p is a nonnegative $C(\mathbb{R}^n)$ function which is c-positive with $\Omega = \mathbb{R}^n$ and*

$$\int_0^\infty r\phi(r)\,dr < \infty, \tag{22}$$

where $\phi(r) \equiv \max_{|x|=r} p(x)$. Then Eq.(1) has a positive entire explosive solution provided f satisfies Condition (4).

Proof: By virtue of Proposition 2.1.1 there exist nonnegative solutions v_k and w_k to the following boundary value problems

$$\begin{aligned} \Delta v_k &= p(x)g_1(v_k), \quad |x| < k, \qquad (23)\\ v_k(x) &= \infty, \quad |x| = k,\\ \Delta w_k &= p(x)g_2(w_k), \quad |x| < k, \qquad (24)\\ w_k(x) &= \infty, \quad |x| = k. \end{aligned}$$

Since

$$\begin{aligned} \Delta v_k &= p(x)g_1(v_k) \le p(x)g_2(v_k), \quad |x| < k, \qquad (25)\\ v_k(x) &= w_k(x), \quad |x| = k, \qquad (26) \end{aligned}$$

by the maximum principle we have

$$w_k \le v_k, \quad |x| \le k. \tag{27}$$

Using an analogous upper/lower solution argument as in Proposition 2.2.1 we have that there exists a nonnegative solution u_k to the boundary value problem

$$\begin{aligned} \Delta u_k &= p(x)f(u_k), \quad |x| < k, \\ u_k(x) &= \infty, \quad |x| = k, \end{aligned} \tag{28}$$

with $w_k \le u_k \le v_k$. If we can show that the sequence u_k is uniformly bounded and equicontinuous on any bounded domain, then the Ascoli-Arzela Theorem guarantees that u_k has a convergent subsequence on that domain. To do this, we first note that without loss of generality we may assume $0 \in \Omega$. Now, consider the ball $B(0,1) \subseteq \Omega = \mathbb{R}^n$ centered at zero with radius one. Notice that $u_k \le v_k$, and that v_k is decreasing. Thus, $v_k \searrow v$. Thus, we have that $u_k \le v$ on $B(0,1)$ for all k. Hence, the sequence u_k is uniformly bounded on the closed ball $\overline{B(0,1)}$ centered at zero with radius one. We also have that u_k is a solution to Eq.(28) on $B(0,1)$, and $u_k \in C^2(B(0,1))$. Thus, by Theorem 1.2.4, we have the gradient bound

$$\sup_{|x|<2} d_x |\nabla u_k(x)| \le C(\sup_{|x|<2} |u_k| + \sup_{|x|<2} d_x^2 |p(x)f(u_k(x))|), \tag{29}$$

where $C = C(n)$ and $d_x = dist(x, \partial B(0,2))$. Furthermore, since $d_x \ge 1$ we have the following result

$$\sup_{|x|<1} |\nabla u_k(x)| \le \sup_{|x|<1} d_x |\nabla u_k(x)| \le \sup_{|x|<2} d_x |\nabla u_k(x)|. \tag{30}$$

Hence, from (29) and (30) the sequence u_k is also equicontinuous on $\overline{B(0,1)}$. It now follows, by the Ascoli-Arzela Theorem, that there exists a subsequence u_{k_i} which converges to a nonnegative function u^1, that is, $u_{k_i} \to u^1$ on the ball $B(0,1) \subseteq \Omega$.

Now, consider the subsequence u_{k_i} on the ball $B(0,2) \subseteq \Omega = \mathbb{R}^n$ centered at 0 with radius two. It is clear that the subsequence u_{k_i} is uniformly bounded on $\overline{B(0,2)}$. Furthermore, u_{k_i} is a solution to (28), with k replaced by k_i, on $B(0,2)$,

and therefore $u_{k_i} \in C^2(B(0,2))$. Thus, we have the gradient bound

$$\sup_{|x|<3} d_x |\nabla u_{k_i}(x)| \leq C(\sup_{|x|<3} |u_{k_i}| + \sup_{|x|<3} d_x^2 |p(x) f(u_{k_i}(x))|), \tag{31}$$

where $C = C(n)$ and $d_x = dist(x, \partial B(0,3))$. Again, since $d_x \geq 1$ we have

$$\sup_{|x|<2} |\nabla u_{k_i}(x)| \leq \sup_{|x|<2} d_x |\nabla u_{k_i}(x)| \leq \sup_{|x|<3} d_x |\nabla u_{k_i}(x)|. \tag{32}$$

Thus, the subsequence u_{k_i} is also equicontinuous on $\overline{B(0,2)}$. Hence, by the Ascoli-Arzela Theorem we have that there exists a subsequence $u_{k_{i_j}}$ which converges to a nonnegative function u^2, that is, $u_{k_{i_j}} \to u^2$ on the ball $B(0,2) \subseteq \Omega$.

Continuing this line of reasoning, we have that there exists positive solutions u^3, u^4, u^5, ..., on the balls $B(0,3)$, $B(0,4)$, $B(0,5)$, ..., respectively. Furthermore we note that

$$u^1 = u^2 = u^3 = u^4 = u^5 = ..., \tag{33}$$

on the ball $B(0,1)$, and in general on the ball $B(0,m)$ we have

$$u^m = u^{m+1} = ...$$

We now need to show that the sequence of solutions $\{u^i\}_{i=1}^{\infty}$ converges on $\Omega = \mathbb{R}^n$ to some function u. A standard bootstrap argument will then show that u is a solution of Eq.(1). Notice that on the intersection of balls $B(0,i)$ centered at 0 with radius i the sequence of solutions u^j are equal for all $i \leq j$. Thus, for any $x \in B(0,k)$, $\lim_{i\to\infty} u^i(x) = u^k(x) \equiv u(x)$ for all $|x| \leq k$. Hence, $u^i(x) \to u(x)$, for all $x \in \mathbb{R}^n$. A standard bootstrap argument, as outlined in the proof of Theorem 2.1.2, will show that $u(x)$ is indeed a solution to (1). Lastly, we show that our solution u is an explosive solution to (1) on $\Omega = \mathbb{R}^n$. Recall that the sequence of solutions $u_k(x) = \infty$

for $|x| = k$ (i.e., the sequence of solutions u_k are explosive.). Thus, the sequence of solutions $u^k(x) = \infty$ for $|x| = k$. It then follows that $u(x) = \infty$ as $|x| \to \infty$. Hence, u is a positive entire explosive solution to (1), completing the proof.

■

We now wish to extend this result to somewhat arbitrary, unbounded domains which the following theorem demonstrates.

Theorem 2.2.3. *Suppose Ω is an unbounded domain in $\mathbb{R}^n$, $n \geq 3$, with compact C^2 boundary and suppose there exists a sequence of bounded domains $\{\Omega_k\}$, each with smooth boundary, such that $\Omega_k \subseteq \Omega_{k+1}$ for all $k = 1, 2, ...$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Suppose p is a nonnegative $C(\overline{\Omega})$ function which is also c-positive on Ω. Let*

$$\phi(r) = \max\{p(x) : |x| = r,\ x \in \Omega\},$$

and assume that ϕ satisfies inequality (22). Then, Eq.(1) has a positive explosive solution provided f satisfies Condition (4).

Proof: We replace the functions v_k and w_k in the proof of Theorem 2.2.2 with the solutions to

$$\begin{aligned} \Delta v_k &= p(x) g_1(v_k), \qquad x \in \Omega_k, \\ v_k(x) &= k, \qquad x \in \partial\Omega_k, \end{aligned}$$

and

$$\begin{aligned} \Delta w_k &= p(x) g_2(w_k), \qquad x \in \Omega_k, \\ w_k(x) &= k, \qquad x \in \partial\Omega_k, \end{aligned}$$

for each k. The proof now follows an analogous approach to that of Theorem 2.2.2. We omit the details.

$\blacksquare$

As in [12] we would like to establish the partial converse of Theorem 2.2.3. More specifically, assuming the same hypothesis as Theorem 2.2.3 (especially concerning the function p), we would like to prove that f satisfies Condition (4) whenever Eq.(1) has a nonnegative large solution on Ω. Although this result is true for $f(s) = s^\gamma$ and $\Omega = \mathbb{R}^n$ (see [11]), we have been unable to establish it for general f except for functions p which have quite specific decay rates, which we shall prove in the corollary below. However, before proving the corollary, we first provide Theorems 2.2.4 and 2.2.6, two crucial "partial converses" to Theorem 2.2.3. Theorem 2.2.4 is specifically important in that it, combined with Theorem 2.2.2, demonstrates that, for functions f such as $f(u) = u^\gamma$, (1) has a nonnegative entire large solution if and only if $\gamma > 1$, that is, provided inequality (22) holds. This further extends the results of [12] and [11]. More generally, for functions f for which Condition (19) is equivalent to inequality (4) (e.g., $f(s) = s^\gamma$), this establishes the desired converse to Theorem 2.2.3 when $\Omega = \mathbb{R}^n$.

Theorem 2.2.4. *Suppose the function p satisfies the hypothesis of Theorem* 2.2.2 *including inequality* (22)*. If Eq.*(1)*, with f replaced with g_1, has a nonnegative entire explosive solution, then f satisfies inequality* (19)*.*

Proof: From [12] we have that since g_1 satisfies inequality (4) there exists a nonnegative entire explosive solution v_1 of $\Delta v_1 = p(x)g_1(v_1)$. From [12] we have that since Eq.(3) has a nonnegative entire explosive solution that g_1 satisfies

$$\int_1^\infty \frac{ds}{g_1(s)} < \infty.$$

Hence, since $g_1 \le f$ it follows that

$$\int_1^\infty \frac{ds}{f(s)} \le \int_1^\infty \frac{ds}{g_1(s)} < \infty. \tag{34}$$

That is, f satisfies condition (19). This completes the proof.

$\blacksquare$

Lemma 2.2.5. *Suppose there exists a nonnegative function h that is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ such that $0 \le \phi(r) \le h^2(r)$ for all $r \ge 0$ and h satisfies one of the following:*

(a) there exists a constant C such that $r^{n-1}h(r) \le C$ for all $r \ge 0$; or
(b) $\lim_{r\to\infty} r^{n-1}h(r) = \infty$ and $\int_0^\infty h(r)\,dr < \infty$.

If v is a nonnegative entire explosive radial solution of $\Delta v = \phi(r)f(v)$, then f satisfies inequality (4).

Remark. Amazingly enough we are able to use an identical proof of Lair (see Lemma 2 of [12]) to show our results. We give the details here for completeness.

Proof: Following Osserman [15], we note that v satisfies $(r^{n-1}v')' = r^{n-1}\phi(r)f(v)$ and multiply this expression by $r^{n-1}u'$. Since $v' \ge 0$, we get

$$\begin{aligned}\left[(r^{n-1}v')^2\right]' &= 2r^{2n-2}\phi(r)\frac{d}{dr}\int_0^{v(r)} f(s)\,ds \\ &\le 2r^{2n-2}h^2(r)\frac{d}{dr}\int_0^{v(r)} f(s)\,ds. \qquad (35)\end{aligned}$$

If h satisfies (a), then (35) produces

$$\left[(r^{n-1}v')^2\right]' \le 2C^2\frac{d}{dr}\int_0^{v(r)} f(s)\,ds.$$

Integrating this over $[0,r]$ yields ($\psi(s) \equiv \int_0^s f(t)\,dt$):

$$(r^{n-1}v')^2 \le 2C^2\int_{v(0)}^{v(r)} f(s)\,ds \le 2C^2\psi(v(r)).$$

Taking the square root of both sides and rearranging terms gives

$$\frac{d}{dr}\int_{v(1)}^{v(r)} [\psi(s)]^{-1/2}\,ds \le 2Cr^{1-n}.$$

Integrating this over $[1, R]$ produces

$$\int_{v(1)}^{v(R)} [\psi(s)]^{-1/2}\, ds \le \frac{2C}{n-2}(1 - R^{2-n}) \le \frac{2C}{n-2}.$$

Letting $R \to \infty$ we have that f satisfies (4).

If h satisfies (b), then we integrate (35) directly. After integrating by parts on the right side and dividing by r^{2n-2}, we get

$$(v')^2 = 2h^2(r) \int_0^{v(r)} f(s)\, ds - 2r^{2-2n} \int_0^r (s^{2n-2}h^2(s))'\psi(v(s))\, ds. \tag{36}$$

The second term on the right side of this equation may be written as

$$2h^2(r) \frac{-\int_0^r (s^{2n-2}h^2(s))'\psi(v(s))\, ds}{r^{2n-2}h^2(r)}. \tag{37}$$

For this ratio, we apply L'Hospital's rule (which is allowed since the denominator diverges to infinity) to get

$$\lim_{n\to\infty} \frac{-\int_0^r (s^{2n-2}h^2(s))'\psi(v(s))\, ds}{r^{2n-2}h^2(r)} = \lim_{n\to\infty} -\psi(v(r)) = -\infty,$$

since $v(r) \to \infty$ as $r \to \infty$. Thus, the expression (37) is negative for large r. Using this fact in (36) produces (for sufficiently large R):

$$(v')^2 \le 2h^2(r)\psi(v(r)) \quad \mathit{for}\ r \ge R,$$

Hence,

$$\int_R^r [\psi(v(s))]^{-1/2} u'(s)\, ds \le 2 \int_R^r h(s)\, ds,$$

from which we get

$$\int_{v(R)}^{v(r)} [\psi(s)]^{-1/2}\, ds \leq 2 \int_R^r h(s)\, ds.$$

Letting $r \to \infty$ and observing that the right side, by hypothesis, converges to a real number, we have that f satisfies (4). This completes the proof.

$\blacksquare$

It is important to note that the previous lemma considers only the case where the solution v is assumed to be a nonnegative entire explosive radial solution of $\Delta v = \phi(r)f(v)$. In order to establish the following theorem, which, in turn, will assist us in establishing our partial converse to Theorem 2.2.3, we first need a lemma that considers the case where v is a nonnegative explosive radial solution of $\Delta v = \phi(r)f(v)$ on a bounded domain.

Lemma 2.2.6. *Suppose there exists a nonnegative function h that is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ such that $0 \leq \phi(r) \leq h^2(r)$ for all $r \geq 0$ and h satisfies Condition (a) of Lemma* 2.2.5. *If v is a nonnegative explosive radial solution of $\Delta v = \phi(r)f(v)$ on $|x| \leq R$, then f satisfies inequality* (4).

Remark. We note here that the proof follows identically to first part of Lemma 2.2.5 with some minor changes. We highlight the changes here.

Proof: As in Lemma 2.2.5, we have for $\psi(s) \equiv \int_0^s f(t)\, dt$,

$$\frac{d}{dr} \int_{v(1)}^{v(r)} [\psi(s)]^{-1/2}\, ds \leq 2Cr^{1-n}.$$

Integrating this over $[1, R - \varepsilon]$, for $\varepsilon > 0$, produces

$$\int_{v(1)}^{v(R-\varepsilon)} [\psi(s)]^{-1/2}\, ds \leq \frac{2C}{n-2}(1 - (R-\varepsilon)^{2-n}) \leq \frac{2C}{n-2}.$$

Now, taking the limit as $\varepsilon \to 0^+$ we have

$$\lim_{\varepsilon \to 0^+} \int_{v(1)}^{v(R-\varepsilon)} [\psi(s)]^{-1/2} \, ds \leq \frac{2C}{n-2}.$$

Thus,

$$\int_1^\infty [\psi(s)]^{-1/2} \, ds \leq \frac{2C}{n-2} < \infty,$$

completing the proof.

■

Theorem 2.2.7. *Let p be a c-positive, nonnegative $C(\overline{\Omega})$ function of Ω, an unbounded domain satisfying the hypothesis of Theorem* 2.2.3*. Let ϕ be defined as in Theorem* 2.2.3 *and satisfy the hypothesis of Lemma* 2.2.5*, except part (b). If u is a nonnegative explosive solution of Eq.*(1)*, then f satisfies inequality* (4)*.*

Proof: Without loss of generality, we may assume, that $0 \in \Omega$ and use an argument inspired by Osserman [15]. We first show that for any $a \geq 0$ the equation

$$\Delta \tilde{u} = \phi(r) f(\tilde{u}) \tag{38}$$

has a nonnegative solution with initial values $\tilde{u}(0) = a$ and $\tilde{u}'(0) = 0$ valid in some interval $0 \leq r \leq r_0$. We note here that in [15] this result follows easily from having f' continuous everywhere. We now write (38) in the integral form

$$\tilde{u}(r) = a + \int_0^r s^{1-n} \int_0^s t^{n-1} \phi(t) f(\tilde{u}(t)) \, dt \, ds, \tag{39}$$

and for the moment replace the function f by f_M defined by $f_M(s) = f(s)$ for $0 \leq s \leq M$ and $f_M(s) = f(M)$ for $s \geq M$. Currently we have not specified a particular value for M as it will be chosen momentarily. We now apply a standard iteration procedure by letting $u_0 = a$ and generating a sequence $\{u_k\}$ in which u_k is

determined from u_{k-1} by

$$u_k(r) = a + \int_0^r s^{1-n} \int_0^s t^{n-1}\phi(t) f_M(u_{k-1}(t))\, dt\, ds. \tag{40}$$

Since

$$u_k'(r) = r^{1-n} \int_0^r t^{n-1}\phi(t) f_M(u_{k-1}(t))\, dt \geq 0,$$

we get $u_k'(r) \geq 0$ for all $k = 1, 2, ...$ and $r \geq 0$ so that (40) gives

$$0 \leq u_k(r) \leq a + \Phi(r)\Psi_M$$

where

$$\Phi(r) = \int_0^r s^{1-n} \int_0^s t^{n-1}\phi(t)\, dt\, ds,$$

and

$$\Psi_M \equiv \max_{0 \leq s \leq M} f_M(s).$$

We then have, noting first that $\lim_{r\to 0} \Phi(r) = 0$, that we may choose M to be any number larger than a and choose $r_0 > 0$ near zero so that $a + \Phi(r_0)\Psi_{M_{r_0}} \leq M$. Then, an induction argument may now be used to show that $u_k(r) \leq M$ for all $k = 1, 2, ...$ and $0 \leq r \leq r_0$. That is, u_k is uniformly bounded for all $k = 1, 2, ...$ and $0 \leq r \leq r_0$. Furthermore, we have $u_k'(r) \leq \Phi(r_0)\Psi_{M_{r_0}} \leq M$ which implies the sequence u_k is equicontinuous on $[0, r_0]$. Hence, by the Ascoli-Arzela Theorem there exists a convergent subsequence u_{k_i} which converges to a nonnegative function $\tilde{u}$ on $[0, r_0]$. The function $\tilde{u}$ is a solution of (39) for $0 \leq r \leq r_0$ where f is replaced by f_M. However, since u_k is bounded above by M for all k, it follows that the subsequence

u_{k_i} is also bounded above by M for all $i = 1, 2, ...,$ and so $\tilde{u} \leq M$. Thus, $\tilde{u}$ is a solution of (39) on $[0, r_0]$.

Now, let $[0, R)$ be the maximum interval in which $\tilde{u}$ exists. Since

$$\tilde{u}'(r) = r^{1-n} \int_0^r s^{n-1} \phi(s) f(\tilde{u}(s))\, ds \geq 0,$$

we get $\tilde{u}'(r) \geq 0$ for all $r \geq 0$, and hence if $R < \infty$, we must have $\lim_{r \to R} \tilde{u}(r) = \infty$. Then, Lemma 2.2.6 implies that f must satisfy (4) since $\tilde{u}$ would be a nonnegative explosive solution of (41) on $|x| \leq R$. Thus, let us now suppose $R = \infty$. If $\lim_{r \to \infty} \tilde{u} = \infty$ we have that f must satisfy (4) due to Lemma 2.2.5, excluding part (b). Suppose, for a contradiction, $R = \infty$ and $\lim_{r \to \infty} \tilde{u}(r) = M_0 < \infty$. If this were the case then the maximum principle would imply that $\tilde{u}(|x|) \leq u(x)$ for all $x \in \Omega$. This would imply that $a = \tilde{u}(0) \leq u(0)$ for any $a > 0$, which is ridiculous. Thus, $\lim_{r \to \infty} \tilde{u}(r) = \infty$ and hence, f must satisfy (4). This completes the proof.

■

As previously stated, if the function p satisfies, in addition to the hypothesis of Theorem 2.2.3, a sufficiently rapid decay condition on at infinity, then the condition on the function f given by (4) is both necessary and sufficient to ensure the existence of an explosive solution of (1) on Ω. We now establish this result.

Corollary 2.2.8. *Suppose Ω is an unbounded domain that satisfies the hypothesis of Theorem* 2.2.3. *Let p be a nonnegative $C(\overline{\Omega})$ function which is also c-positive, and assume that there exists a constant K such that for $|x|$ large and $x \in \Omega$, $p(x) \leq K|x|^{-\alpha}$, $\alpha \geq 2n-2$. Then, a necessary and sufficient condition for Eq.*(1) *to have a nonnegative explosive solution on Ω is that f satisfy inequality* (4).

Proof: Sufficiency is clear from Theorem 2.2.3. In order to prove necessity, we need only show that the function ϕ, defined as in Theorem 2.2.3, satisfies the hypothesis of Lemma 2.2.5, except part (b), so that Theorem 2.2.7 may be applied.

Letting $h(r) = Kr^{-\alpha/2}$ for large r, we have that part (a) of Lemma 2.2.5 is satisfied for $\alpha \geq 2n - 2$. Thus, we now invoke Theorem 2.2.7, establishing the corollary.

$\blacksquare$

III. Conclusion

3.1 Conclusion

We began our research by attempting to establish existence of solutions to the non-monotone semilinear elliptic equation

$$\Delta u \;=\; p(x)f(u), \qquad x \in \Omega \subseteq \mathbb{R}^n,\; n \geq 3,$$

where the function f is nonnegative on $[0, \infty)$ and satisfies the inequality $g_1 \leq f \leq g_2$. In our case, the function f was not monotone on $[0, \infty)$. This particular assumption on f had not been considered in previous problems of this type. In proving the existence of solutions we have further extended the results of Lair [12], who considered the same equation with the additional condition that f be nondecreasing.

Our first case considered the existence of solutions to Eq.(1) on a bounded domain $\Omega \subseteq \mathbb{R}^n$. For our first result, Proposition 2.1.1, we established the existence of a nonnegative classical solution v to Eq.(9) on a bounded domain Ω by cleverly using an upper/lower solution argument along with inequality (2). We then showed, in Theorem 2.1.2, that there exists a nonnegative explosive solution u on a bounded domain provided the function g_1 satisfied the integral growth condition given by inequality (4). Establishing this result was done by employing a more difficult variation of the upper/lower solution argument in order to produce a sequence of solutions $\{u_k\}$ that were monotone and bounded above. Then, applying the standard bootstrap argument showed that u was a classical solution of (1). Along with this existence result we also showed, in Theorem 2.1.2, that if inequality (5) holds then Eq.(1) has no nonnegative explosive solution. This particular result was established by contradiction.

We further set out to establish the existence of solutions to (1) on unbounded domains Ω, including $\Omega = \mathbb{R}^n$. Unlike the previous case we required an additional,

asymptotic, condition on the function p in order to establish results on Ω similar to those of Theorem 2.1.2. We then showed, in Theorem 2.2.2, that Eq.(1) had a positive entire explosive solution u provided the function f satisfied inequality (4). The proof of Theorem 2.2.2 turned out to be quite difficult since the sequence of solutions $\{u_k\}$ generated from an upper/lower solution argument analogous to the one used in Proposition 2.1.2 were not monotone, and we were not able to construct a monotone sequence as done in Theorem 2.1.2. Instead we were able to establish the existence of a positive entire explosive solution u to (1) by showing that the sequence of solutions $\{u_k\}$ were both uniformly bounded and equicontinuous. With both conditions met we then applied the Ascoli-Arzela Theorem to show the sequence u_k converged. This method was then applied to successive subsequences of the sequence $\{u_k\}$ on successively larger domains (balls) which, eventually, covered the whole space $\mathbb{R}^n$. Another bootstrap argument showed that u was a classical solution of (1). This result further extends the results of [12].

After establishing Theorem 2.2.2 we then extended our results by showing that on some unbounded domain Ω in $\mathbb{R}^n$, Eq.(1) had a positive explosive solution u provided f satisfied inequality (4). This result was easily established with some minor changes, and applying the same method as in Theorem 2.2.2. At this point we had generalized the results of [12] for our case. However, it turned out that for our later results we did not have to use the functions g_1 or g_2 to establish existence of solutions. This was quite astounding since establishing the existence of solutions to Eq.(1) when the function f is not monotone had never been done before.

Our first result where g_1, g_2 were no longer necessary was in Lemma 2.2.5, where we showed that if u was a nonnegative entire explosive radial solution of $\Delta u = \phi(r)f(u)$, then f satisfied inequality (4). We then established a similar lemma on a bounded domain. It was interesting that both lemmas were established using an identical method of Lair (see Lemma 2 of [12]), especially since f was not monotone. With these two lemmas, along with Theorem 2.2.7, which established a partial

converse to Theorem 2.2.3, we showed that a necessary and sufficient condition for Eq.(1) to have a nonnegative explosive solution on Ω is that f satisfy (4). This result, given by Corollary 2.2.8, is the most significant result of this thesis since it generalizes the main result of [12].

In this thesis we have provided a useful generalization to the results of [12] in which the condition on the function f is relaxed to allow f to be non-monotone. This, in turn, allows Eq.(1) to be used for a more diverse range of problems that may require f to be a non-monotone function, and in our case, bounded between two monotone functions.

3.2 Further Work

Although we established many important results, we did encounter two problems in establishing results for the function f being non-monotone. In Lemma 2.2.1, we showed that for g_1 satisfying inequality (4), f satisfied (19). We believe this result can be done for f satisfying (4), that is, if f satisfies inequality (4) then f satisfies (19). We also encountered a similar problem with generalizing Theorem 2.2.4. In this problem we established our results for Eq.(1), with f replaced by g_1, having a nonnegative entire explosive solution then f satisfied (19). It would be of great use to provide the same result without having to replace f by g_1 in (1).

If more time were permitted we would have liked to analyze our problem numerically. A numerical approach may assist in solving Lemma 2.2.1 and Theorem 2.2.4. Another interesting problem would be to establish existence of solutions to systems of equations of this type. Lastly, there still remains the open problem of whether the existence of solutions to Eq.(1) can be established for f non-monotone and with the conditions on g_1 and g_2 weakened to allow for an even more generalized result. Lastly, an even greater challenge would be to establish the existence of solutions to (1) for f non-monotone and without the condition $g_1 \leq f \leq g_2$. Although

we are not sure if this can even be done , it would pose a challenging problem and most likely done at the doctoral level.

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Vita

Second Lieutenant Zachary J. Proano was born in Reading, Pennsylvania. He graduated from Exeter Twp Senior High School in June 2000. From there he went to Grand Forks, North Dakota to attended The University of North Dakota (UND) where he graduated Cum Laude with a Bachelor of Science in Mathematics in May 2004. Lieutenant Proano received his Commission through Air Force ROTC Detachment 610 Eagle Wing II at UND.

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14. ABSTRACT
We consider the problem $\Delta u = p(x) f(u)$ in some domain Ω in Euclidean n-space. Such problems arise in the study of steady state diffusion type problems, the subsonic motion of a gas, the electric potential in some bodies, and Riemannian geometry. We consider the semilinear elliptic equation $\Delta u = p(x) f(u)$, on a domain Ω in Euclidean n-space, $n \geq 3$, where f is a nonnegative function which vanishes at the origin and satisfies $g_1 \leq f \leq g_2$ where g_1, g_2 are both nonnegative, nondecreasing functions which also vanish at the origin, and p is a nonnegative continuous function with the property that any zero of p is contained in a bounded domain Ω such that p is positive on its boundary. For Ω bounded, we show that a nonnegative solution u satisfying $u(x) \to \infty$ as $x \to \partial\Omega$ exists provided the function $\psi(s) \equiv \int_0^s f(t)dt$ satisfies $\int_1^\infty [\psi(s)]^{-1/2} ds < \infty$. For Ω unbounded, we show that a similar results holds where $u(x) \to \infty$ as $|x| \to \infty$ within Ω and $u(x) \to \infty$ as $x \to \partial\Omega$ if $p(x)$ decays to zero rapidly as $|x| \to \infty$.

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